Risky Arbitrage, Asset Prices, and Externalities

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Current Version, June 2006†

Abstract

We introduce a no-risky-arbitrage price condition (NRAP) for asset market models allowing both unbounded short sales and externalities such as trading volume. We then demonstrate that NRAP is sufficient for the existence of competitive equilibrium in the presence of externalities. Moreover, we show that if all risky arbitrages are utility increasing, then NRAP characterizes competitive equilibrium in the presence of externalities.

JEL classifications: C62, D50

KEYWORDS: Risky Arbitrage, Competitive Equilibrium, Viable Asset Prices

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†We are indebted to an anonymous referee for careful reading and helpful comments on an earlier version of this paper. Page and Wooders are especially grateful to CERMSEM and EUREQua for their support and hospitality which made possible our collaboration.

1 Introduction

In competitive asset markets trading volume influences investors' expectations of future asset returns, and thus, influences equilibrium asset prices. The influence of trading externalities, such as trading volume on equilibrium asset prices, is brought about by a process of arbitrage elimination that characterizes informationally efficient asset markets. While there have been numerous papers investigating the connections between arbitrage and equilibrium asset prices in asset market models with unbounded short sales, with one exception there has been no work on the connections between arbitrage and asset prices in models with short sales where trading externalities are taken into account.[1]

In this paper, we introduce a no-risky-arbitrage price condition, NRAP, for models allowing both trading externalities and unbounded short sales, and demonstrate that NRAP is sufficient, and in some cases necessary, for the existence of competitive equilibrium in the presence of externalities. In empirical studies of financial markets, available information may well include both prices and volumes of net trades. Thus, it is important to have characterizations depending on prices and observable data. In fact, our research follows the fundamental work of Hammond (1983) for asset market models and Werner (1987) for general equilibrium models.

In a risky arbitrage, an agent sells an existing portfolio and buys a utility nondecreasing alternative portfolio for a net cost less than or equal to zero. Whether a particular pair of transactions (selling a portfolio and buying another) constitutes a risky arbitrage thus depends on the agent's preferences as well as asset prices and, in the presence of externalities, each agent's preferences in turn depend directly on the trades of other agents. In its most potent form, a risky arbitrage is *utility increasing* and generates a net cost less than or equal to zero.[2] Here, we formalize the notion of risky arbitrage in an asset market model with trading externalities and short sales and introduce a condition on asset prices that rules out risky arbitrage for all agents. Given the close connection between agent preferences and risky arbitrage, NRAP is essentially an assumption concerning the degree of homogeneity in

[1] See Le Van, Page, and Wooders (2001).

[2] In a *riskless* arbitrage, an agent sells an existing portfolio and buys a replicating portfolio (i.e., an alternative portfolio having the same returns in all states of nature) for a net cost less than or equal to zero. Thus, a riskless arbitrage is a special case of a risky arbitrage. In its most potent form, a riskless arbitrage generates a positive amount of money upfront - or put differently, a riskless arbitrage can be carried out via a pair of trades having a net cost strictly less than zero.

agents' preferences.

The intuition behind our results is straightforward: with sufficient homogeneity, even if trading externalities are present and unbounded short sales are allowed, if NRAP is satisfied an agent will be unable to carry out a risky arbitrage because no one will be willing to take the other side of the transaction. However, with externalities, carrying out a transaction may perturb the arbitrage opportunities for all agents and lead to further changes, even reversing the desirability of the initial transaction. Such considerations make formulation of NRAP delicate.

Besides being sufficient for existence of equilibrium, whenever all risky arbitrages are utility increasing then NRAP is also necessary for existence of equilibrium. Thus, in asset markets with externalities and short sales in which all risky arbitrages are utility increasing, NRAP characterizes competitive equilibrium. Moreover, for any given level of the externalities, NRAP ensures existence of demand functions.

In the literature, no-risky-arbitrage (NRA) conditions for asset market models without trading externalities fall into three broad categories: (i) *conditions on net trades*, for example, Hart (1974), Page (1987), Nielsen (1989), Page, Wooders, and Monteiro (2000), and Allouch (2002); (ii) *conditions on prices*, for example, Grandmont (1970,1977), Green (1973), Hammond (1983), and Werner (1987); (iii) *conditions on the set of utility possibilities (namely compactness)*, for example Brown and Werner (1995) and Dana, Le Van, and Magnien (1999). In Le Van, Page, and Wooders (2001) an NRA condition on net trades is introduced for models with trading externalities and short sales - a condition that reduces to the condition of Hart (1974) if no externalities are present - and it is shown that the net trades NRA condition is sufficient for existence. Since NRAP reduces to the condition of Werner (1987) if no externalities are present and enables proof of existence of equilibrium in the presence of externalities, our research continues the prior work. We further relate our condition to prior conditions by showing that, if all risky arbitrages are utility increasing, then NRAP and the NRA net trades condition are equivalent, and both characterize competitive equilibrium.

In an economic model similar to the model presented here, but without externalities, Dana, Le Van, and Magnien (1999) have shown that compactness of the set of utility possibilities is sufficient for the existence of competitive equilibrium. However, in the presence of externalities compactness of utility possibilities, as a condition limiting arbitrage, is not sufficient for existence.

2 An Economy with Trading Externalities

Let $(X_j, \omega_j, u_j(\cdot,\cdot))_{j=1}^n$ denote an unbounded exchange *economy* with trading externalities. In the economy $(X_j, \omega_j, u_j(\cdot,\cdot))_{j=1}^n$ each agent j has *choice set* $X_j \subset R^L$ and *endowment* $\omega_j \in X_j$. The j^{th} agent's preferences, defined over $X := \prod_{j=1}^n X_j$, are specified via a *utility function* $u_j(\cdot,\cdot) : X_j \times X_{-j} \to R$, where $X_{-j} := \prod_{i \neq j} X_i$. Note that for all agents j, $X = X_j \times X_{-j}$. Let x_{-j} denote a typical element of X_{-j}. Often it will useful to denote the elements in X by (x_j, x_{-j}).

The set of *rational allocations* is given by

$$\begin{aligned} A = \{(x_1, ..., x_n) \in X : \textstyle\sum_{j=1}^n x_j = \sum_{j=1}^n \omega_j \\ andforeachj, u_j(x_j, x_{-j}) \geq u_j(\omega_j, x_{-j})\}. \end{aligned} \tag{1}$$

We will denote by A_{-j} the projection of A onto X_{-j}.

For each $(x_j, x_{-j}) \in \prod_{j=1}^n X_j$, the *preferred set* is given by

$$P_j(x_j, x_{-j}) := \{x \in X_j : u_j(x, x_{-j}) > u_j(x_j, x_{-j})\}, \tag{2}$$

while the weakly preferred set is given by

$$\widehat{P}_j(x_j, x_{-j}) := \{x \in X_j : u_j(x, x_{-j}) \geq u_j(x_j, x_{-j})\}. \tag{3}$$

We will maintain the following assumptions on the economy $(X_j, \omega_j, u_j(\cdot,\cdot))_{j=1}^n$: For each $j = 1, ..., n$,

$$[A-1] \quad \begin{cases} X_j isclosedandconvex, and \omega_j \in int X_j, \\ where ``int" \ denotes ``interior". \end{cases}$$

$$[A-2] \quad \begin{cases} Foreach (x_j, x_{-j}) \in X, u_j(\cdot, x_{-j}) isquasi-concaveon X_j, \\ and u_j(\cdot,\cdot) iscontinuouson X_j \times X_{-j}. \end{cases}$$

$$[A-3] \quad \begin{cases} Foreach (x_j, x_{-j}) \in A, \ P_j(x_j, x_{-j}) \neq \emptyset, \\ andcl P_j(x_j, x_{-j}) = \widehat{P}_j(x_j, x_{-j}). \end{cases}$$

Note that in [A-1] we do not assume that consumption sets, X_j, are bounded. Also, note that given [A-2], for all $(x_j, x_{-j}) \in X$ the preferred set $P_j(x_j, x_{-j})$ is nonempty and convex, while the weakly preferred set $\widehat{P}_j(x_j, x_{-j})$ is nonempty, closed and convex. Finally, note that [A-3] implies that there is *local nonsatiation at rational allocations.*

Given prices $p \in R^L$, the cost of a consumption vector $x = (x_1, ..., x_L)$ is $\langle p, x\rangle = \sum_{\ell=1}^{L} p_\ell \cdot x_\ell$. The *budget set* is given by[3]

$$B_j(p, \omega_j) = \{x \in X_j : \langle p, x\rangle \leq \langle p, \omega_j\rangle\}. \tag{4}$$

Without loss of generality we can assume that prices are contained in the unit ball

$$\mathcal{B} := \{p \in R^L : \|p\| \leq 1\}.$$

An *equilibrium* for the economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ is an $(n+1)$-tuple of vectors $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$ such that

(i) $(\overline{x}_1, ..., \overline{x}_n) \in A$ (the allocation is feasible);

(ii) $\overline{p} \in \mathcal{B}\backslash\{0\}$ (prices are in the unit ball and not all prices are zero); and

(iii) for each j,

(a) $\langle \overline{p}, \overline{x}_j\rangle = \langle \overline{p}, \omega_j\rangle$ (budget constraints are satisfied), and

(b) $\overline{x}_j \in B_j(\overline{p}, \omega_j)$ and $P_j(\overline{x}_j, \overline{x}_{-j}) \cap B_j(\overline{p}, \omega_j) = \emptyset$ (i.e., $\overline{x}_j$ maximizes $u_j(x_j, \overline{x}_{-j})$ over $B_j(\overline{p}, \omega_j)$).

We provide an example illustrating our model in application to an asset market. This example will be further developed later in the paper.

Example: Part 1, An Asset Market with Trading Externalities.
Consider an agent j who seeks to form a portfolio $x_j = (x_{1j}, \ldots, x_{Lj})$ of L risky assets so as to maximize his expected utility given by

$$u_j(x_j, x_{-j}) = \int_{R^L} U_j(\langle x_j, r\rangle) d\mu_j(r|x_{-j}).$$

Here, x_{ij} denotes the number of (perfectly divisible) shares of asset i held in the j^{th} agent's portfolio x_j, and r_i denotes the return on asset i, i.e., the i^{th} component of the asset return vector $r \in R_+^L$.[4] The inner product of the portfolio vector x_j and the asset return vector r, denoted by

$$\langle x_j, r\rangle = \sum_{i=1}^{L} x_{ij} r_i,$$

[3] The restriction of the budget set to be a subset of the consumption set entails no losss of substance or generality.

[4] R_+^L denotes the nonnegative orthant of R^L. Thus, here we are assuming that all asset returns are nonnegative or, equivalently, that all assets carry limited liability.

gives the return generated by portfolio x_j if the realized asset return vector is r. Note that because short sales are allowed, $\langle x_j, r\rangle$ can be negative. The function

$$U_j(\cdot) : R \to R$$

is the j^{th} agent's utility function defined over end-of-period wealth. The probability measure $\mu_j(\cdot|x_{-j})$ defined over Borel subsets of asset returns represents the j^{th} agent's subjective probability beliefs concerning end-of-period asset returns conditioned by the $(n-1)$-tuple, x_{-j}, of portfolios held by other agents.

Denote by $S[\mu_j(\cdot|x_{-j})]$ the support of the conditional probability measure $\mu_j(\cdot|x_{-j})$, and by $K(x_{-j})$ the smallest convex cone containing $S[\mu_j(\cdot|x_{-j})]$. Finally, let $K^+(x_{-j})$ denote the positive dual cone of $K(x_{-j})$, that is, let

$$K^+(x_{-j}) := \left\{ y \in R^L : \langle y, r\rangle \geq 0 \forall r \in K(x_{-j}) \right\}.$$

Note that any vector of net trades y contained in $K^+(x_{-j})$ generates a nonnegative return with probability 1. Thus, trading in any direction $y \in K^+(x_{-j})$ is without downside risk.

Assume the following:

(a-1) For each agent $j = 1, 2, \ldots, n$, the utility function $U_j(\cdot) : R \to R$ is concave and increasing.

(a-2) For each agent $j = 1, 2, \ldots, n$, the mapping,

$$x_{-j} \to \mu_j(\cdot|x_{-j}),$$

is continuous in the topology of weak (or narrow) convergence of probability measures.

(a-3) For all rational allocations $(x_j, x_{-j}) \in A$ and for all agents $j = 1, 2, \ldots, n$, $S[\mu_j(\cdot|x_{-j})] \cap R^L_+ \backslash \{0\} \neq \emptyset$.

(a-4) For all $x_{-j} \in X_{-j}$ and for all agents $j = 1, 2, \ldots, n$,

$$S[\mu_j(\cdot|x_{-j})] \subseteq C$$

for some bounded set $C \subset R^L_+$.

(a-5) For all agents $j = 1, 2, \ldots, n$, the portfolio choice set X_j is closed and convex with initial portfolio $\omega_j \in intX_j$, and for all $(x_j, x_{-j}) \in X$, $y \in K^+(x_{-j})$ implies that $x_j + y \in X_j$.

In words, assumption (a-3) means that at rational allocations each agent believes that some asset will generate a positive return with a positive probability. Assumption (a-5) means that given any configuration of starting portfolios $(x_j, x_{-j}) \in X$, agent j can alter (or rebalance) his starting portfolio x_j via net trades $y \in K^+(x_{-j})$ (i.e., via a no-downside-risk portfolio) without violating portfolio feasibility (i.e., without violating his constraint set X_j). Note that together assumptions (a-1), (a-3), and (a-5) imply that agents' expected utility preferences satisfy assumption [A-3] (local nonsatiation) while assumptions (a-1) and (a-2) imply that agents' expected utility preferences satisfy assumptions [A-2] (quasiconcavity and continuity).

3 Risky Arbitrage and NRAP

We begin by recalling a few basic facts about recession cones (see Section 8 in Rockafellar (1970)). Let X be a convex set in R^L. The *recession cone* $0^+(X)$ corresponding to X is given by

$$0^+(X) = \{y \in R^L : x + \lambda y \in X \; forall \lambda \geq 0 and x \in X\}. \tag{5}$$

If X is also closed, then the set $0^+(X)$ is a closed convex cone containing the origin. Moreover, if X is closed, then $x + \lambda y \in X$ for *some* $x \in X$ and all $\lambda \geq 0$ implies that $x' + \lambda y \in X$ for *all* $x' \in X$ and all $\lambda \geq 0$. Thus, if X is closed, then we can conclude that $y \in 0^+(X)$ if for some $x \in X$ and all $\lambda \geq 0$, $x + \lambda y \in X$.

(Risky Arbitrage):

A vector of net trades $y_j \in R^L$ is a risky arbitrage for agent j if there exists a sequence

$$\left\{x^k\right\}_k = \left\{(x_j^k, x_{-j}^k)\right\}_k = \left\{(x_1^k, \ldots, x_n^k)\right\}_k \subset X$$

such that

$$forall k, u_j(x_j^k, x_{-j}^k) \geq u_j(\omega_j, x_{-j}^k),$$

and

$$\begin{gathered} y_j = \lim_k t^k x_j^k \\ for some sequence \left\{t^k\right\}_k of positive real numbers \\ such that t^k \downarrow 0. \end{gathered}$$

Denote by R_j the set of all risky arbitrages for agent j.

Let $(x_j, x_{-j}) \in X$ satisfy $x_j \in \widehat{P}_j(\omega_j, x_{-j})$. If $y_j \in 0^+\widehat{P}_j(\omega_j, x_{-j})$, then y_j is a risky arbitrage. Thus, any vector of net trades contained in the recession cone of the weakly preferred set $\widehat{P}_j(\omega_j, x_{-j})$ is a risky arbitrage for agent j. To see this, note that $y_j = \lim_k t^k x_j^k$ with $t^k = \frac{1}{k+1}$ and $x_j^k = \omega_j + (k+1)y_j$ and $x_j^k \in \widehat{P}_j(\omega_j, x_{-j})$ since $y_j \in O^+(\widehat{P}_j(\omega_j, x_{-j}))$. We have then

$$\begin{gathered} \left\{(x_j^k, x_{-j}^k)\right\}_k = \{(\omega_j + (k+1)y_j, x_{-j})\}_k \subset X, \\ and \\ y_j = \lim_k t^k x_j^k, \\ where t^k \downarrow 0, \\ and where for all k, \\ u_j(x_j^k, x_{-j}^k) = u_j(\omega_j + (k+1)y_j, x_{-j}) \geq u_j(\omega_j, x_{-j}). \end{gathered}$$

Example: Part 2, Recession Cones and Risky Arbitrage.

Continuing our asset market example, let

$$\begin{gathered} s^j(+) := \lim_{c\to\infty} \frac{dU_j(c)}{dc} \\ and \\ s^j(-) := \lim_{c\to-\infty} \frac{dU_j(c)}{dc} \end{gathered}$$

denote the asymptotic values of derivatives of the jth agent's utility function $U_j(\cdot)$ in the positive and negative directions respectively and add to our list of assumptions the following assumption:

(a-6) for each agent j the ratio of the asymptotic derivatives $s^j := \frac{s^j(+)}{s^j(-)}$ is zero.

The ratio s^j is an asymptotic measure of risk tolerance. It inversely measures the concavity of $U_j(\cdot)$ as c goes from $-\infty$ to ∞. We adopt the convention that $s^j = 0$ when $s^j(-) = \infty$. Note that under assumption (a-1) (see Example, Part 1), $0 \leq s^j \leq 1$ for all agents j. If $s^j = 1$ then the agent is risk neutral and thus has the highest level of asymptotic risk tolerance. If $s^j < 1$ the agent is risk averse and therefore has a lower level of asymptotic risk tolerance. In particular, if $s^j = 0$ then the agent has the lowest level of asymptotic risk tolerance. It is easy to verify that $s^j = 0$ for all constant absolute risk aversion utility functions.

By Lemma 5.2 in Page (1987), if $s^j = 0$, then

$$O^+(\widehat{P}_j(\omega_j, x_{-j})) = K^+(x_{-j}),$$

where, as in Example: Part 1, $K^+(x_{-j})$ is the positive dual cone of $K(x_{-j})$.[5]

[5]Recall that $K(x_{-j})$ is the convex cone generated by the support $S[\mu_j(\cdot|x_{-j})]$ of the conditional probability measure $\mu_j(\cdot|x_{-j})$.

We now have our main result characterizing risky arbitrage.

(Characterization of Risky Arbitrage)

Let $(X_j, \omega_j, u_j(\cdot,\cdot))_{j=1}^n$ be an economy with trading externalities satisfying assumptions [A-1]-[A-3]. The following statements are equivalent:

1. A vector of net trades $y_j \in R^L$ is a *risky arbitrage* for agent j.

2. There exists a sequence $\left\{(x_j^k, x_{-j}^k)\right\}_k \subset X$ such that

$$y_j \in 0^+ \left(\lim \widehat{P}_j(\omega_j, x_{-j}^k)\right).$$

The limit, $\lim \widehat{P}_j(\omega_j, x_{-j}^k)$, of the sequence of closed sets $\left\{\widehat{P}_j(\omega_j, x_{-j}^k)\right\}_k$ in part 2 of Theorem 1 is taken with respect to Painleve-Kuratowski convergence (see, for example section B.II p. 15 in Hildenbrand (1974) for definitions and properties). Before we prove Theorem 1, we provide the following Lemma.

Let $(X_j, \omega_j, u_j(\cdot,\cdot))_{j=1}^n$ be an economy with trading externalities satisfying assumptions [A-1]-[A-3]. Let $\left\{(x_j^k, x_{-j}^k)\right\}_k \subset X$ be a sequence such that for all j and k, $x_j^k \in \widehat{P}_j(\omega_j, x_{-j}^k)$. Also let $\left\{t^k\right\}_k$ be a sequence of positive real numbers with $t^k \downarrow 0$. If $(y_1, \ldots, y_n)$ is a cluster point of the sequence $\left\{(t^k x_1^k, \ldots, t^k x_n^k)\right\}_k$, then there exists a subsequence $\left\{(t^{k'} x_1^{k'}, \ldots, t^{k'} x_n^{k'})\right\}_{k'}$ such that for all j, $y_j \in 0^+ \left(\lim \widehat{P}_j(\omega_j, x_{-j}^{k'})\right)$.

Proof: (Lemma 3) Without loss of generality, assume that

$$(y_1, \ldots, y_n) = \lim_k (t^k x_1^k, \ldots, t^k x_n^k).$$

From Hildenbrand (1974), Proposition 1, p. 16, there exists a converging subsequence $\left\{\left(\widehat{P}_1(\omega_1, x_{-1}^{k'}), \ldots, \widehat{P}_n(\omega_n, x_{-n}^{k'})\right)\right\}_{k'}$ of $\left\{\left(\widehat{P}_1(\omega_1, x_{-1}^k), \ldots, \widehat{P}_n(\omega_n, x_{-n}^k)\right)\right\}_k$. Observe that for all j, $\lim \widehat{P}_j(\omega_j, x_{-j}^{k'})$ is convex (see Danzig, Folkman, and Shapiro (1967), p. 521) and nonempty since it contains ω_j. Also note that $(y_1, \ldots, y_n) = \lim_{k'} (t^{k'} x_1^{k'}, \ldots, t^{k'} x_n^{k'})$.

Now let $x_j^* \in \lim \widehat{P}_j(\omega_j, x_{-j}^{k'})$ and let t be *any* positive number. By the definition of $\lim \widehat{P}_j(\omega_j, x_{-j}^{k'})$, there exists a sequence $\left\{x_j^{*k'}\right\}_{k'}$ such that $x_j^{*k'} \to x_j^*$, as $k' \to \infty$, and for all k', $x_j^{*k'} \in \widehat{P}_j(\omega_j, x_{-j}^{k'})$. Since $\widehat{P}_j(\omega_j, x_{-j}^{k'})$ is convex for k' large enough so that $t^{k'} t \leq 1$,

$$(1 - t^{k'} t) x_j^{*k'} + t^{k'} t x_j^{k'} \in \widehat{P}_j(\omega_j, x_{-j}^{k'}).$$

But

$$(1-t^{k'}t)x_j^{*k'} + t^{k'}tx_j^{k'} \to x_j^* + ty_j \in \lim \widehat{P}_j(\omega_j, x_{-j}^{k'}).$$

Thus, $y_j \in 0^+ \left(\lim \widehat{P}_j(\omega_j, x_{-j}^{k'})\right)$. ■

Proof: (Theorem 3) (1) $\Rightarrow$ (2). Let y_j be a risky arbitrage for agent j and let $\left\{(x_j^k, x_{-j}^k)\right\}_k \subset X$ be such that

$$\begin{gathered} forallk, u_j(x_j^k, x_{-j}^k) \geq u_j(\omega_j, x_{-j}^k), and \\ y_j = \lim_k t^k x_j^k fort^k \downarrow 0. \end{gathered}$$

Then either $\left\{\left\|x_j^k\right\|\right\}_k$ is bounded and $y_j = 0$ or $\left\{\left\|x_j^k\right\|\right\}_k$ is unbounded and from the Lemma, $y_j \in 0^+ \left(\lim \widehat{P}_j(\omega_j, x_{-j}^{k'})\right)$ for some subsequence $\left\{(x_j^{k'}, x_{-j}^{k'})\right\}_{k'}$.

(2) $\Rightarrow$ (1). Conversely, let $y_j \in 0^+ \left(\lim \widehat{P}_j(\omega_j, x_{-j}^k)\right)$ for some sequence

$$\left\{(x_j^k, x_{-j}^k)\right\}_k \subset X.$$

Let $\{\lambda^m\}_m$ be a sequence of real numbers such that $\lambda^m \uparrow \infty$. Since

$$y_j \in 0^+ \left(\lim \widehat{P}_j(\omega_j, x_{-j}^k)\right),$$

we have $\omega_j + \lambda^m\ y_j \in \lim \widehat{P}_j(\omega_j, x_{-j}^k)$ for all m. Let $\varepsilon > 0$. For each m there exists k_m and $x_j^{k_m} \in \widehat{P}_j(\omega_j, x_{-j}^{k_m})$ such that

$$\left\|\omega_j + \lambda^m y_j - x_j^{k_m}\right\| < \varepsilon.$$

This implies that

$$\left\|\frac{\omega_j}{\lambda^m} + y_j - \frac{x_j^{k_m}}{\lambda^m}\right\| < \frac{\varepsilon}{\lambda^m}.$$

Letting $m \to \infty$, we conclude that $\frac{x_j^{k_m}}{\lambda^m} \to y_j$. Because $x_j^{k_m} \in \widehat{P}_j(\omega_j, x_{-j}^{k_m})$ for all m and because $\frac{1}{\lambda^m} \to 0$, y_j is a risky arbitrage for agent j. ■

(Closedness of the set of Risky Arbitrages)

Let $(X_j, \omega_j, u_j(\cdot,\cdot))_{j=1}^n$ be an economy with trading externalities satisfying assumptions [A-1]-[A-3]. Then, for each agent j, the set of risky arbitrages, R_j, is closed.

Proof: Let $\{y^\nu\}_\nu \subset R_j$ be a sequence of arbitrages for the jth agent such that

$$y^\nu \to y.$$

We want to show that $y \in R_j$. By our characterization of risky arbitrage, we have for each ν, a sequence $\left\{(x_j^{k,\nu}, x_{-j}^{k,\nu})\right\}_k \subset X$ such that $y^\nu \in 0^+\left(\lim_k \widehat{P}_j(\omega_j, x_{-j}^{k,\nu})\right)$. Let $\varepsilon > 0$ and let $\{\lambda_m\}_m$ be a sequence of real numbers such that $\lambda_m \uparrow \infty$. For all m and ν there exists a positive integer $k(m,\nu)$ such that

$$(i) \left\|\omega_j + \lambda_m y^\nu - x_j^{k(m,\nu),\nu}\right\| \leq \varepsilon$$
$$and$$
$$(ii) x_j^{k(m,\nu),\nu} \in \widehat{P}_j(\omega_j, x_{-j}^{k(m,\nu),\nu}).$$

From (i) it follows that

$$\left\|\frac{\omega_j}{\lambda_m} + y^\nu - \frac{x_j^{k(m,\nu),\nu}}{\lambda_m}\right\| \leq \frac{\varepsilon}{\lambda_m}.$$

Therefore,

$$\left\|\frac{x_j^{k(m,\nu),\nu}}{\lambda_m}\right\| \leq \frac{\varepsilon}{\lambda_m} + \left\|\frac{\omega_j}{\lambda_m}\right\| + \|y^\nu\|,$$

and hence $\left\{\frac{x_j^{k(m,\nu),\nu}}{\lambda_m}\right\}_{(m,\nu)}$ is bounded. In particular, the sequence $\left\{\frac{x_j^{k(n,n),n}}{\lambda_n}\right\}_n$ is bounded. Let z_j be a cluster point of this sequence. Then z_j is a risky arbitrage and $z_j = y$. ■

(The No-Risky-Arbitrage Price Condition, NRAP):

(1) $p \in R^L$ is a NRAP price for agent j if $\langle p, y_j \rangle > 0$ for all nonzero risky arbitrages $y_j \in R_j \backslash \{0\}$.

(2) Let S_j denote the jth agent's set of NRAP prices. The economy $(X_j, \omega_j, u_j(\cdot,\cdot))_{j=1}^n$ with trading externalities satisfies NRAP if

$$\cap_j S_j \neq \emptyset.$$

Note that the set of NRAP prices S_j is a convex cone. More importantly, note that any price vector $p \in \cap_j S_j$ assigns a positive value to the risky arbitrages of any agent. Thus, if p is a no-risky-arbitrage price, then each agent's demand correspondence is nonempty at p no matter what consumption vectors are chosen by other agents (see section 6 below on viable prices, and in particular, see part 1 of Theorem 6.1).

If there exists a *pointed* closed convex cone $C \subset R^L$ such that each agent's set of risky arbitrages, R_j, is contained in C, then NRAP is satisfied.[6] In particular, by classical separation arguments, for C a pointed closed convex cone there exists a nonzero vector $p \in R^L$ such that $\langle p, y \rangle > 0$ for all nonzero $y \in C$ - and thus, $\langle p, y_j \rangle > 0$ for all nonzero risky arbitrages $y_j \in R_j$. Conversely, since the risky arbitrage sets, R_j, are cones, if NRAP is satisfied then given prices p contained in $\cap_j S_j$, there exists $\alpha > 0$ such that for all j, R_j is contained in the pointed convex cone C given by

$$C = \left\{ y \in R^L : \langle p, y \rangle > \alpha \left\| y \right\| \right\}.$$

Example: Part 3, The Existence of a Closed Pointed Cone Containing All Risky Arbitrages.

Assume that

(a-7) each agent j has conditional probability beliefs, $\mu_j(\cdot|x_{-j})$, concerning asset returns such that for some closed convex cone, K_j, with nonempty interior

$$K_j = K(x_{-j}) forall (x_j, x_{-j}) \in X,$$

where again $K(x_{-j})$ is the convex cone generated by the support $S[\mu_j(\cdot|x_{-j})]$ of $\mu_j(\cdot|x_{-j})$.

It is important to note that the invariance of the cones $K(x_{-j})$ with respect to x_{-j} (i.e., with respect to the trades of other agents) does not imply that conditional probability beliefs are invariant with respect to x_{-j}. Moreover, nonemptiness of the interior of K_j implies that no asset returns are perfectly correlated.

In light of Example: Part 2, we can conclude that if assumptions (a-1)-(a-6) are satisfied and if assumption (a-7) holds, then for all $(x_j, x_{-j}) \in X$

$$O^+(\widehat{P}_j(\omega_j, x_{-j})) = K_j^+,$$

where K_j^+ is the positive dual cone of K_j. Moreover, under (a-1)-(a-7), for each agent j the set of risky arbitrages R_j is equal to K_j^+.

By Proposition 3 in Page (1996), the *jth* agent's set of NRAP prices, S_j, is equal to the interior of K_j (denoted $intK_j$), and thus, NRAP is satisfied if and only if

$$\cap_j intK_j \neq \emptyset$$

[6]Some authors take "pointed" to mean only that the cone contains the origin. We take a pointed cone to be one which contains the origin and does not contain a line.

(i.e., a price vector p is a vector of no-risky-arbitrage prices if and only if $p \in \cap_j intK_j$). Finally, by Proposition 5 in Page (1996), under assumptions (a-1)-(a-7), $\cap_j intK_j \neq \emptyset$ if and only if

$$\sum_{j=1}^{n} y_j = 0 with y_j \in R_j for all j implies that y_j = 0 for all j.(*)$$

Under (a-1)-(a-7) it is easy to show that condition (*), the no-mutually-compatible-arbitrages condition, holds if and only if there is a pointed closed convex cone $C \subset R^L$ such that each agent's set of risky arbitrages, R_j, is contained in C.

One of the main implications of NRAP is compactness of the set of rational allocations. This implication is a key ingredient in our proof of existence of a competitive equilibrium.

(NRAP implies compactness of rational allocations):

Let $(X_j, \omega_j, u_j(\cdot,\cdot))_{j=1}^{n}$ be an economy with trading externalities satisfying assumptions [A-1]-[A-3]. If the economy satisfies NRAP then the set of rational allocations, A, is compact.

Proof: Since A is closed, we have just to prove that A is bounded. Suppose not. Then there is a sequence $\left\{(x_1^k, \ldots, x_n^k)\right\}_k \subset A$ such that $\sum_j \left\| x_j^k \right\| \to \infty$ as $k \to \infty$. Letting $t^k := \frac{1}{\sum_j \| x_j^k \|}$, we have for some subsequence $\left\{(x_1^{k'}, \ldots, x_n^{k'})\right\}_{k'}$,

$$\begin{gathered}(t^{k'} x_1^{k'}, \ldots, t^{k'} x_n^{k'}) \to (y_1, \ldots, y_n) \\ with \\ \textstyle\sum_j \|y_j\| = 1.\end{gathered}$$

We have $(y_1, \ldots, y_n) \neq 0$ and by definition, $(y_1, \ldots, y_n)$ is a risky arbitrage. By NRAP, there exists a price vector $p \in \cap_j S_j$ such that

$$\langle p, y_j \rangle > 0 for j = 1, 2, \ldots, n.$$

Thus,

$$\sum_j \langle p, y_j \rangle = \left\langle p, \sum_j y_j \right\rangle > 0.$$

But now we have a contradiction because

$$\begin{gathered}\textstyle\sum_j t^{k'} x_j^{k'} = \sum_j t^{k'} \omega_j for all k \\ and therefore \\ \textstyle\sum_j t^{k'} x_j^{k'} \to \sum_j y_j = 0.\end{gathered}$$

■

4 Existence of Equilibrium

4.1 Existence for Bounded Economies with Externalities

We begin by defining a k-bounded economy,

$$(X_{kj}, \omega_j, u_j(\cdot,\cdot))_{j=1}^n, \tag{6}$$

In the k-bounded economy, the j^{th} agent's consumption set is

$$X_{kj} := X_j \cap B_k(\omega_j), \tag{7}$$

where $B_k(\omega_j)$ is a closed ball of radius k centered at the agent's endowment, ω_j.

Define

$$X_k := \prod_{j=1}^n X_{kj}.$$

The set of k-bounded rational allocations is given by

$$\begin{aligned} A_k = \{(x_1, ..., x_n) \in X_k : \textstyle\sum_{j=1}^n x_j = \sum_{j=1}^n \omega_j \\ and for each j, u_j(x_j, x_{-j}) \geq u_j(\omega_j, x_{-j})\}. \end{aligned} \tag{8}$$

By Theorem 3.6 above, if the original economy $(X_j, \omega_j, u_j(\cdot,\cdot))_{j=1}^n$ satisfies NRAP, then the set of rational allocations is compact. Thus, there exists some integer k^* such that for all $k \geq k^*$, $A_k = A$.

An *equilibrium* for the k-bounded economy, $(X_{kj}, \omega_j, u_j(\cdot))_{j=1}^n$, is an $(n+1)$-tuple of vectors $(x_1^k, \ldots, x_n^k, p^k)$ such that

(i) $(x_1^k, \ldots, x_n^k) \in A_k$, (the allocation is feasible);

(ii) $p^k \in \mathcal{B}\backslash\{0\}$ (prices are in the unit ball and not all prices are zero); and

(iii) for each j,

(a) $\left\langle p^k, x_j^k \right\rangle = \left\langle p^k, \omega_j \right\rangle$ (budget constraints are satisfied), and

(b) $x_j^k \in B_{kj}(p^k, \omega_j)$ and $P_{kj}(x_j^k, x_{-j}^k) \cap B_{kj}(p^k, \omega_j) = \emptyset$ (i.e., x_j^k maximizes $u_j(x_j, x_{-j}^k)$ over $B_{kj}(p^k, \omega_j)$).

Here,

$$P_{kj}(x_j^k, x_{-j}^k) := P_j(x_j^k, x_{-j}^k) \cap X_{kj},$$
$$\text{and}$$
$$B_{kj}(p^k, \omega_j) := B_j(p^k, \omega_j) \cap X_{kj}.$$

We now have our main existence result for bounded economies.

(Existence of an equilibrium for k-bounded economies)

Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ be an economy with trading externalities satisfying assumptions [A-1]-[A-3] and let k^* satisfy the condition that $A_{k^*} = A$. Then for all $k \geq k^*$ the k-bounded economy,

$$(X_{jk}, \omega_j, u_j(\cdot, \cdot))_{j=1}^n,$$

has an equilibrium, $(x_1^k, \ldots, x_n^k, p^k)$, with

$$p^k \in \mathcal{B}_u := \left\{ p \in R^L : \|p\| = 1 \right\}.$$

Proof: Because the original economy $(X_j, \omega_j, u_j(\cdot, \cdot))_{j=1}^n$ satisfies local nonsatiation at rational allocations (i.e., assumption [A-3]), for all k *larger than* k^* such that $A_k = A$ for $k \geq k^*$, the k-bounded economy $(X_{jk}, \omega_j, u_j(\cdot, \cdot))_{j=1}^n$ also satisfies local nonsatiation at rational allocations. Thus it follows from Florenzano (2003), chapter 2, that for k larger than k^*, the k-bounded economy $(X_{jk}, \omega_j, u_j(\cdot, \cdot))_{j=1}^n$ has an equilibrium. ■

4.2 Existence for Unbounded Economies with Externalities

Our main existence result for unbounded economies with externalities is the following:

(Existence for unbounded economies with externalities)

Let $(X_j, \omega_j, u_j(\cdot, \cdot))_{j=1}^n$ be an economy with trading externalities satisfying assumptions [A-1]-[A-3]. If the economy satisfies NRAP, then the economy has an equilibrium, $(\overline{x}_1, \ldots, \overline{x}_n, \overline{p})$, with

$$\overline{p} \in \mathcal{B}_u := \left\{ p \in R^L : \|p\| = 1 \right\}.$$

.

Proof: For each k larger than k^* such that $A_k = A$ for $k \geq k^*$, the k-bounded economy $(X_{jk}, \omega_j, u_j(\cdot, \cdot))_{j=1}^n$ has an equilibrium

$$(x_1^k, \ldots, x_n^k, p^k) = (x^k, p^k) \in A_k \times \mathcal{B}_u = A \times \mathcal{B}_u.$$

Since $A \times \mathcal{B}_u$ is compact, we can assume without loss of generality that

$$(x_1^k, \ldots, x_n^k, p^k) \to (\overline{x}_1, \ldots, \overline{x}_n, \overline{p}) \in A \times \mathcal{B}_u.$$

Moreover, since for all j and k, $\left\langle p^k, x_j^k \right\rangle = \left\langle p^k, \omega_j \right\rangle$, we have for all j, $\langle \overline{p}, \overline{x}_j \rangle = \langle \overline{p}, \omega_j \rangle$.

Let $u_j(x_j, \overline{x}_{-j}) > u_j(\overline{x}_j, \overline{x}_{-j})$. Then, for $k > k^*$ sufficiently large, $x_j \in X_{jk}$ and $u_j(x_j, x_{-j}^k) > u_j(x_j^k, x_{-j}^k)$ which implies that $\left\langle p^k, x_j \right\rangle > \left\langle p^k, \omega_j \right\rangle$. Thus, in the limit $\langle \overline{p}, x_j \rangle \geq \langle \overline{p}, \omega_j \rangle$. Hence, $(\overline{x}_1, \ldots, \overline{x}_n, \overline{p})$ is a quasi-equilibrium. Since for all j, $\omega_j \in intX_j$ (see [A-1]) and since utility functions are continuous (see [A-2]), in fact, $(\overline{x}_1, \ldots, \overline{x}_n, \overline{p})$ is an equilibrium. ■

5 Necessary and Sufficient Conditions for Existence

We begin by introducing the following uniformity conditions:

$$[A-4] \quad \begin{cases} If y_j \in R_j \backslash \{0\}, then \\ forall (x_j, x_{-j}) \in A, u_j(x_j + y_j, x_{-j}) > u_j(x_j, x_{-j}). \end{cases}$$

By assumption [A-4] all risky arbitrages are utility increasing provided that the starting point for the risky arbitrage is a rational allocation.

Now we have our main result on necessary and sufficient conditions for existence.

(NRAP $\Leftrightarrow$ existence of equilibrium)

Let $(X_j, \omega_j, u_j(\cdot, \cdot))_{j=1}^n$ be an economy with trading externalities satisfying assumptions [A-1]-[A-4]. Then the following statements are equivalent:

1. $(X_j, \omega_j, u_j(\cdot, \cdot))_{j=1}^n$ satisfies NRAP.
2. $(X_j, \omega_j, u_j(\cdot, \cdot))_{j=1}^n$ has an equilibrium.

Proof: By Theorem 4.2, we know that (1) $\Rightarrow$ (2). So we need only establish that (2) $\Rightarrow$ (1). Let $(\overline{x}, \overline{p})$ be an equilibrium and for some agent j suppose that $y_j \in R_j \backslash \{0\}$ is a risky arbitrage. By [A-4], $u_j(\overline{x}_j + y_j, \overline{x}_{-j}) > u_j(\overline{x}_j, \overline{x}_{-j})$. Because $(\overline{x}, \overline{p})$ is an equilibrium $\langle \overline{p}, \overline{x}_j + y_j \rangle > \langle \overline{p}, \omega_j \rangle = \langle \overline{p}, \overline{x}_j \rangle$. Thus, $\langle \overline{p}, y_j \rangle > 0$. ■

6 Viable Prices and Externalities

In this section we extend Kreps' (1981) notion of viable prices to exchange economies with externalities and establish the relationship between NRAP and viable prices. To begin, consider the problem

$$\max \left\{ u_j(x_j, x_{-j}) : x_j \in \widehat{P}_j(\omega_j, x_{-j}) and \, \langle p, x_j \rangle \leq \langle p, \omega_j \rangle \right\},$$

where $x_{-j} \in X_{-j}$ is given. We say that price vector p is *viable* for agent j if this problem has a solution for any $x_{-j} \in X_{-j}$. Thus, p is viable for agent j if agent $j's$ demand correspondence is nonempty at p no matter what consumption vector $x_{-j} \in X_{-j}$ is chosen by other agents. Consider now the following strengthening of assumption [A-4],

$$[A-4]^* \quad \left\{ \begin{array}{l} If y_j \in R_j \backslash \{0\}, then \\ forall (x_j, x_{-j}) \in X, u_j(x_j + y_j, x_{-j}) > u_j(x_j, x_{-j}). \end{array} \right.$$

By assumption [A-4]* all risky arbitrages are utility increasing starting at any $(x_j, x_{-j}) \in X$.

(NRAP and viable prices)

Let $(X_j, \omega_j, u_j(\cdot, \cdot))_{j=1}^n$ be an economy with trading externalities satisfying assumptions [A-1]-[A-3]. Then the following statements are true:

1. If p is an NRAP price for agent j, then p is viable for agent j.
2. If assumption [A-4]* also holds, then if p is viable for agent j, then p is an NRAP price for agent j.

Proof: (1) Since $u_j(\cdot, \cdot)$ is continuous, it suffices to prove that the set

$$\left\{ x \in R^L : x \in \widehat{P}_j(\omega_j, x_{-j}) and \, \langle p, x \rangle \leq \langle p, \omega_j \rangle \right\}$$

is bounded. If not, let $\left\{ x^k \right\}_k$ be an unbounded sequence which satisfies

$$\begin{array}{c} x^k \in \widehat{P}_j(\omega_j, x_{-j}) \\ and \\ \left\langle p, x^k \right\rangle \leq \langle p, \omega_j \rangle \, forallk. \end{array}$$

Let y be a cluster point of the sequence $\left\{ \frac{x^k}{\|x^k\|} \right\}_k$. Then y is a risky arbitrage and $\langle p, y \rangle \leq 0$, a contradiction since p is an NRA price vector for agent j.

(2) Conversely, let p be viable and assume [A-4]* holds. Let $\overline{x}$ solve the problem

$$\max\left\{u_j(x_j, x_{-j}) : x_j \in \widehat{P}_j(\omega_j, x_{-j}) and \, \langle p, x_j\rangle \leq \langle p, \omega_j\rangle\right\},$$

and suppose $y \neq 0$ is a risky arbitrage. By [A-4]*

$$u_j(\overline{x} + y, x_{-j}) > u_j(\overline{x}, x_{-j}).$$

We have

$$\langle p, \omega_j + y\rangle \geq \langle p, \overline{x} + y\rangle$$
$$and$$
$$\langle p, \overline{x} + y\rangle > \langle p, \omega_j\rangle \, implies \, \langle p, y\rangle > 0.$$

■

By Theorem 6.1, if the economy satisfies [A-1]-[A-3] then the NRAP condition guarantees the existence of a nonempty set of viable prices for the economy (i.e., for all agents), and thus, guarantees the existence of demand functions over the set of viable prices. In addition, by Theorem 6.1, if all risky arbitrages are utility increasing starting at any $(x_j, x_{-j}) \in X$ (i.e., if [A-4]* holds), then the existence of demand functions guarantees the existence no-risky-arbitrage prices.

Example: Part 4, The Uniformity Conditions [A-4] and [A-4]*:
Under assumptions (a-1)-(a-7), it follows from Lemma 3 in Page (1996) that each risky arbitrage $y_j \in R_j\backslash\{0\}$ is such that

$$u_j(x_j + y_j, x_{-j}) > u_j(x_j, x_{-j}) forall (x_j, x_{-j}) \in X.$$

Thus, in our asset market example, if assumptions (a-1)-(a-7) hold, then all risky arbitrages $y_j \in R_j\backslash\{0\}$ are utility increasing starting at any $(x_j, x_{-j}) \in X$ (i.e., the uniformity assumption [A-4]* holds - and thus [A-4] holds as well).

7 Conclusions

Externalities are a pervasive feature of economics and, not surprisingly, the subject of ongoing interest in general equilibrium models (see, for example, Florenzano (2003), Bonnisseau (1997), Bonnisseau and Médecin (2001)). Our research contributes to this for a class of models which we feel is of interest and importance – situations where agents may be affected by both prices and trading volume, an indicator of what other agents are doing. Our condition, NRAP forges a link between trading volume and asset prices in markets where arbitrage is possible.

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